through two given points. The ray joining the two given points being a common chord, the conics are determined as soon as its poles are known. Its four poles are the self-corresponding points of the involutory quadratic transformation of which the given lines constitute the fundamental triangle, and by which the given points are interchanged.

These conics will all be real if the involutions determined at A, B, and C

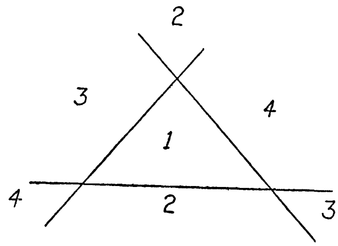


FIG. 5.

have double rays. For this to happen, the rays joining A, for instance, to L and L' must not be divided by b and c. Hence, all the conics will be real if L and L' lie together in one of the four sections into which the given lines divide the plane[1] (Fig. 5). Otherwise, they are all imaginary, since two of the involutions have no double rays.

A SIMPLE METHOD OF CONSTRUCTING THE NORMALS TO A PARABOLA.

By S. G. BARTON, University of Pennsylvania.

We know by Harvey's Theorem that the feet of the three normals to a parabola from any point lie on a circle which passes through the vertex. The converse is also true, the normals whose feet lie on a circle through the vertex are concurrent. For let (d, e) be the center of such a circle. Its equation then is $x^2 + y^2 - 2dx - 2ey = 0$, which intersects the parabola $y^2 = 4px$ in points whose ordinates are given by

$$y^3 - 8p(d - 2p)y - 32p^2 e = 0.$$

Since this lacks the second term the sum of its roots is zero which is the condition for concurrent normals.

The equation giving the ordinates of the feet of the normals through a point (h, k) is

$$y^3 - 4p(h - 2p)y - 8p^2 k = 0.$$

But these equations must be equivalent, hence by equating coefficients $2d - 4p = h - 2p$ and $32e = 8k$; therefore $d = \frac{1}{2}h + p$ and $e = \frac{1}{4}k$.

[1] See Annie Dale Biddle, "Constructive Theory of the Unicursal Plane Quartic by Synthetic Methods," Univ. Calif. Publ. Math., Vol. 1, No. 2.

Hence to construct the normals from the point (h, k), find the feet by constructing a circle whose center is $(\frac{1}{2}h + p, \frac{1}{4}k)$ and radius such that it passes through the vertex of the parabola. The intersections of the circle with the parabola other than the vertex are the feet of the three normals. If the point (h, k) is on the convex side of the evolute of the parabola, the three normals are real and distinct; if on the evolute two are coincident, all three coinciding at the cusp; if on the concave side, two normals are imaginary. Under the same conditions the center of the circle will bear the same relation to the curve $(x - 2p)^3 = \frac{27}{2}py^2$.

If there is difficulty in locating the exact point of intersection, it will often be of aid to recall that the sum of the three ordinates must vanish.

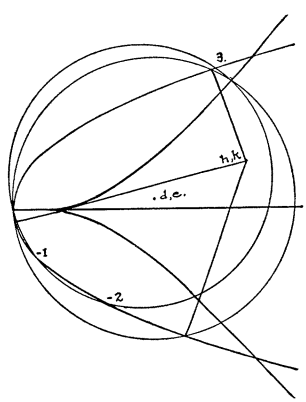
A graphical solution for cubic equations may be based upon the preceding construction for the normals of a parabola, since we have solved the cubic which gives the ordinates of the feet of the normals, namely,

$$y^3 - 8p(d - 2p)y - 32p^2 e = 0.$$

Hence in this manner we can solve any cubic. Let the cubic (which must be transformed so as to lack the second term), be of the form

$$y^3 - ay - b = 0.$$

Equating coefficients we get $d = a/8p + 2p$, $e = b/32p^2$. Hence construct any parabola and describe a circle whose center is (d, e), as determined above, and radius such that it passes through the vertex. Then the ordinates of the inter-



sections, other than the vertex, are the roots of the equation. One parabola only need be drawn, but for convenience the unit of measure should be changed to suit the coefficients. It will usually be found desirable to choose $p = 0.2a$ approximately.

For example to solve the equation $x^3 - 7x - 6 = 0$, let p be 0.5, whence $d = 2.75$ and $e = 0.75$.

This center and the intersections are shown on the figure. Whence we read the roots -1, -2 and 3. If the equation had been $x^3 - 700x - 6{,}000 = 0$, that is, the roots ten times as large, we would take $p = 5$ and the figure would be identical, the unit only being changed. The curve $(x - 2p)^3 = \frac{27}{2}py^2$, which gives the criterion for the nature of the roots is also shown. The roots are: one real and two imaginary, two equal or three real and unequal, according as the center (d, e) is to the left of, on, or to the right of, this curve. The accuracy of the solution varies with the nature of the intersections.

SOME PROPERTIES OF THE NORMALS TO PARABOLAS.

By S. G. BARTON, University of Pennsylvania.

There are in general three normals to a parabola from any point. If the point lies on the evolute of the parabola, two of the normals coincide (all three coincide at the cusp); if on the concave side of the evolute two normals are imaginary; if on the convex side, the three normals are real and distinct. The following properties referring chiefly to normals drawn from points on the evolute were found in an investigation for another purpose.

Consider the parabola $y^2 = 4px$, whose evolute is $27py^2 = 4(x - 2p)^3$. If (a, b) is a point on the evolute, then

(1) The equations of the coincident and single normals are respectively

$$y - b = \sqrt{\frac{a - 2p}{3p}}\,(x - a) \qquad \text{and} \qquad y - b = -\,2\sqrt{\frac{a - 2p}{3p}}\,(x - a)$$

and coördinates of the feet of these normals are

$$\left(\frac{a - 2p}{3},\ -\,2\sqrt{\frac{(a - 2p)p}{3}}\right) \qquad \text{and} \qquad \left(\tfrac{4}{3}(a - 2p),\ 4\sqrt{\frac{(a - 2p)p}{3}}\right).$$

Thus the abscissa of the foot of the single normal is four times that of the double normal, and the ordinate of the foot of the former is minus twice that of the latter. The abscissa of the foot of the double normal is one third of the difference between the abscissæ of the point (a, b) and the cusp of the evolute, and the difference of the abscissæ of the feet is equal to the difference for the point and cusp.

(2) The x intercepts are $\frac{1}{2}(a + 4p)$ and $\frac{1}{3}(4a - 2p)$. Hence the foot of the ordinate of the point (a, b) trisects the distance between the intersections of the normals with the x axis, being nearer the single normal. The difference of the two intercepts is $(a - 2p)$ which is the distance from the foot of the ordinate to the cusp. Thus the single normal cuts the x axis four times as far from the cusp as the double normal. Similarly the foot of the abscissa trisects the distance between the intersections of the normals with the y axis, being nearer the double normal. The x intercept of the double normal is a mean proportional between the abscissæ of the points in which it intersects the parabola.